Dynamics of Gravitomagnetic Charge *

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The physically interesting gravitational analogue of magnetic monopole in electrodynamics is considered in the present paper. The author investigates the field equation of gravitomagnetic matter, and the exact static cylindrically symmetric solution of field equation as well as the motion of gravitomagnetic charge in gravitational fields. Use is made of the mechanism of gravitational Meissner effect, a potential interpretation of anomalous, constant, acceleration acting on the Pioneer 10/11, Galileo and Ulysses spacecrafts is also suggested.

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I. INTRODUCTION

It is well known that the field equation of general relativity under low-motion weak-field approximation is analogous to Maxwell's equation of electromagnetic field. This similarity leads us to consider the following interesting gravitational analogues of electromagnetic phenomena: (i) in electrodynamics a charged particle is acted upon by Lorentz magnetic force, in the same manner, a particle is also acted upon by the gravitational Lorentz force in weak-gravity theory [1,2]. According to the principle of equivalence, further analysis shows that in the non-inertial rotating reference frame, this gravitational Lorentz force is just the fictitious Coriolis force [3]; (ii) there exists the Aharonov-Bohm effect in electrodynamics [4], accordingly, the so-called gravitational Aharonov-Bohm effect, *i.e.*, the gravitational analogue of Aharonov-Bohm effect also exists in theory of gravitation, which is now termed Aharonov-Carmi effect [5–7]; (iii) a particle with intrinsic spin possesses a gravitomagnetic moment of such magnitude that it equals the spin of this particle which can be coupled to gravitomagnetic fields. Mashhoon's spin-rotation coupling is the non-inertial realization of the interaction of gravitomagnetic moment with gravitomagnetic field [8,9]. We think that investigation of general characters of gauge field deserves generalization to that of the gravitational field [10,11]. In this paper, we study another physically interesting phenomenon, *i.e.*, the gravitational analogue of magnetic charge in electrodynamics.

In electrodynamics, electric charge is a Noether charge while its dual charge (magnetic charge) is a topological charge, since the latter is related to the singularity of non-analytical magnetic vector potentials. Magnetic monopole [12] attracts attentions of many physicists in various fields such as gauge field theory, grand unified theory, particle physics and cosmology [13–17]. In the similar fashion, it is also interesting to consider gravitomagnetic charge which is the source of gravitomagnetic field just as mass (gravitoelectric charge) is the source of gravitoelectric field (Newtonian gravitational field). In this sense, gravitomagnetic charge is also termed dual mass. It should be noted that the concept of mass is of no significance for the gravitomagnetic charge, and then it is of interest to investigate the relativistic dynamics and gravitational effects of this topological dual mass. Historically, gravitomagnetic monopole attracts attentions of many authors in both classical dynamics and quantization formulation [18–23]. For historical review one may be referred to the paper by Linden-Bell and Nouri-Zonoz [24].

In literature, to the best of our knowledge, many authors were concerned with *monopole* rather than with *charge*, namely, they considered equations of motion of gravitomagnetic monopoles more than field equations of gravitomagnetic charges/matter. In this paper, the author proposes the field equation of gravitomagnetic matter and investigates

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*The original version of this paper was written in 1999∼2000. In the present paper I would like to discuss some aspects of gravitomagnetic charge in more detail. It won't be submitted elsewhere for publication since most of the subjects in this manuscript are analogous to those presented in one of my papers published in 2002 [$Gen.Rel.Gra.\mathbf{34}(9), 1423-1435(2002)$]. Readers may also be referred to this published paper for a brief review of the essential features of the potential field equations of the hypothetical gravitomagnetic monopole.

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the equation of motion of gravitomagnetic monopole, and then exactly solves the static cylindrically symmetric solution and discusses the motion of a photon in the gravitomagnetic fields. In Sec. V, a potential interpretation of anomalous, constant, acceleration acting on the Pioneer 10/11, Galileo and Ulysses spacecraft [25] is suggested. The relation between the non-analytic property of metric and the topological gravitomagnetic charge is discussed in Sec. VI. In Sec. VII, the author concludes with some remarks.

II. TOPOLOGICAL CHARGE AND FIELD EQUATION OF GRAVITOMAGNETIC MATTER

First we take into consideration the $SO(3)$ non-Abel magnetic monopole proposed by Li *et al.* [26,27], which enables one to consider the topological dual mass in curved spacetime, then we suggest the gravitational field equation of gravitomagnetic charge.

The linear element on the two-dimensional spherical surface is $\mathrm{d}s^2 = r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2$ where the non-vanishing Christoffel symbols $\Gamma^i_{\mu j}$ $(\mu, i, j = 1, 2)$ is given as follows

$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{\cos\theta}{\sin\theta}, \quad \Gamma^1_{22} = -\sin\theta\cos\theta \tag{2.1}$$

and its matrix form $(\Gamma_\theta)^i_j$ and $(\Gamma_\varphi)^i_j$ are therefore

$$(\Gamma_\theta) = \left(\begin{array}{cc} 0 & 0 \\ 0 & \frac{\cos\theta}{\sin\theta} \end{array}\right), \quad (\Gamma_\varphi) = \left(\begin{array}{cc} 0 & -\sin\theta\cos\theta \\ \frac{\cos\theta}{\sin\theta} & 0 \end{array}\right). \tag{2.2}$$

Under the transformation M^{-1}, the basic vector $r\sin\theta\mathrm{d}\varphi$ that varies with θ may be transformed into what is θ-independent, namely,

$$\left(\begin{array}{c} \mathrm{d}\theta \\ \mathrm{d}\varphi \end{array}\right) = M^{-1}\left(\begin{array}{c} r\mathrm{d}\theta \\ r\sin\theta\mathrm{d}\varphi \end{array}\right), \quad M = \left(\begin{array}{cc} r & 0 \\ 0 & r\sin\theta \end{array}\right). \tag{2.3}$$

Accordingly, Γ_μ is transformed into $\tilde{\Gamma}_\mu = M\Gamma_\mu M^{-1} + M\partial_\mu M^{-1}$ and the results are of the form

$$\tilde{\Gamma}_\theta = 0, \quad \tilde{\Gamma}_\varphi = \left(\begin{array}{cc} 0 & -\cos\theta \\ \cos\theta & 0 \end{array}\right) = -\cos\theta X_3 \tag{2.4}$$

with $X_3 = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$ being one of the generators of $SO(3)$ group. There exists the connection between $\tilde{\Gamma}_\mu$ and the potential W^3_μ which may be written as $\tilde{\Gamma}_\mu = gW^3_\mu X_3$ and we therefore obtain $gW^3_\theta = 0, \quad gW^3_\varphi = -\cos\theta$, and the non-vanishing field strengths are written in the forms

$$f^3_{\theta\varphi} = -f^3_{\varphi\theta} = \frac{\partial W^3_\varphi}{\partial\theta} - \frac{\partial W^3_\theta}{\partial\varphi} = \frac{\sin\theta}{g}. \tag{2.5}$$

It follows that the source charge $\tilde{g}$ of field strength $f^3_{\mu\nu}$ is expressed by

$$\tilde{g} = \oint f^3_{\theta\varphi}\mathrm{d}\theta\mathrm{d}\varphi = \frac{4\pi}{g}. \tag{2.6}$$

It should be noted that g is Noether charge while $\tilde{g}$ is topological charge whose existence is related to the singularities of field potentials. The work of Li *et al.* [26] leads us to consider the dual charge associated with spacetime. In fact, although no topological charge $\tilde{g}$ exists seen from in the three-dimensional space, the reason why the two-dimensional surface is curved seen from the observer on the above two-dimensional spherical surface can be equivalently ascribed to the presence of this kind of dual charge. The concept of above topological charge may be readily generalized to that of gravitomagnetic charge.

The dual of Riemann curvature tensor, $R_{\mu\nu\gamma\delta}$, may be defined as follows

$$\tilde{R}_{\mu\nu\gamma\delta} = \epsilon_{\mu\nu}{}^{\alpha\beta}R_{\gamma\delta\alpha\beta} + \epsilon_{\gamma\delta}{}^{\alpha\beta}R_{\mu\nu\alpha\beta} \tag{2.7}$$

with $\epsilon_{\mu\nu}{}^{\alpha\beta}$ being the completely antisymmetric Levi-Civita tensor satisfying its covariant derivative $\epsilon_{\mu\nu}{}^{\alpha\beta}{}_{;\xi} = 0$. It should be noted that $\tilde{R}_{\mu\nu\gamma\delta} \equiv 0$ in the absence of gravitomagnetic matter since no singularities associated with

topological charge exist in the metric functions and therefore Ricci identity still holds. However, once the metric functions possess non-analytic properties in the presence of gravitomagnetic matter (should such exist), the dual curvature tensor is therefore no longer vanishing due to the violation of Ricci identity. The dual curvature scalar that contains only the first derivative and second derivative of field $g_{\mu\nu}$ can be chosen to be $\tilde{R} = -\epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}$. By making use of the variational principle, the dual Einstein's tensor is obtained as follows

$$\begin{aligned} \delta \int_{\Omega} -\sqrt{-g}\epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \mathrm{d}\Omega &= \frac{1}{2} \int_{\Omega} \sqrt{-g} (\tilde{R}_{\mu\nu} - 2\epsilon^{\delta}{}_{\mu}{}^{\alpha\beta} R_{\nu\delta\alpha\beta}) \delta g^{\mu\nu} \mathrm{d}\Omega \\ &= \int_{\Omega} \sqrt{-g} \tilde{G}_{\mu\nu} \delta g^{\mu\nu} \mathrm{d}\Omega \end{aligned} \tag{2.8}$$

with $\tilde{R}_{\mu\nu} = g^{\sigma\tau} \tilde{R}_{\sigma\mu\nu\tau}$. Further calculation yields

$$\tilde{G}_{\mu\nu} = \frac{1}{2} (\epsilon_{\nu}{}^{\alpha\beta\gamma} R_{\mu\alpha\beta\gamma} - \epsilon_{\mu}{}^{\alpha\beta\gamma} R_{\nu\alpha\beta\gamma}), \tag{2.9}$$

which is considered the dual of the Einstein's tensor arising on the left handed side of gravitational field equation of gravitomagnetic matter. Note, however, that once the gravitomagnetic charge is absent in spacetime, $\tilde{G}_{\mu\nu}$ vanishes due to the Ricci identity. But, the non-analytic metric functions caused by the presence of gravitomagnetic charge may result in the violation of Ricci identity and therefore lead to $\tilde{G}_{\mu\nu} \neq 0$. Since the dual Einstein's tensor is an antisymmetric tensor, in what follows we will construct the antisymmetric source tensor of gravitomagnetic charge that is on the right handed side of gravitational field equation. We show that for the Fermion field, the antisymmetric tensors constructed in terms of field ψ and space-time derivatives are of the form

$$K_{\mu\nu} = i\bar{\psi}(\gamma_{\mu}\partial_{\nu} - \gamma_{\upsilon}\partial_{\mu})\psi, \quad H_{\mu\nu} = \epsilon_{\mu\nu}{}^{\alpha\beta} K_{\alpha\beta} \tag{2.10}$$

and regard the linear combination of them as the source tensor in the field equation of gravitomagnetic charge, where γ_{μ}'s denote the general Dirac matrices with respect to x^{μ} and satisfy $\gamma_{\mu}\gamma_{\upsilon} + \gamma_{\nu}\gamma_{\mu} = 2g_{\mu\nu}$. Thus the field equation governing the distribution of gravitation of gravitomagnetic charge may be given as follows

$$\tilde{G}_{\mu\nu} = \kappa_1 K_{\mu\nu} + \kappa_2 H_{\mu\nu} \tag{2.11}$$

with κ_1, κ_2 being the coupling coefficients between gravitomagnetic matter and gravity. One of the advantages of this field equation is that it does not introduce extra tensor potentials when allowing for gravitomagnetic monopole densities and currents. This fact is in analogy with that in electrodynamics, where the equation $\partial_{\nu}\tilde{F}^{\mu\nu} = J^{\mu}_{\mathbf{M}}$ governs the motion of electromagnetic fields produced by magnetic monopoles. Additionally, further analysis shows that the linear combination of the following antisymmetric tensors, $i\bar{\psi}(\gamma_{\mu}\gamma_{\nu} - \gamma_{\upsilon}\gamma_{\mu})\psi$ and $i\epsilon_{\mu\nu}{}^{\alpha\beta}\bar{\psi}(\gamma_{\alpha}\gamma_{\beta} - \gamma_{\beta}\gamma_{\alpha})\psi$ serves as the *cosmological term* in the above gravitational field equation of Fermion field.

It is believed that there would exist formation (and creation) mechanism of gravitomagnetic charge in the gravitational interaction, just as some prevalent theories [16] provide the theoretical mechanism of existence of magnetic monopole in various gauge interactions. Magnetic monopole in electrodynamics and gauge field theory has been discussed and sought after for decades, and the existence of the 't Hooft-Polyakov monopole solution has spurred new interest of both theorists and experimentalists [16,28,29]. Similar to magnetic monopole, gravitomagnetic charge is believed to give rise to such situations. If it is indeed present in universe, it will also lead to significant consequences in astrophysics and cosmology. We emphasize that although it is the classical solution to the field equation as discussed above, this kind of topological gravitomagnetic monopole may arise not as fundamental entities in gravity theory.

III. MOTION OF GRAVITOMAGNETIC CHARGE

According to its gravitational properties, gravitomagnetic charge can be called *dual mass*, and accordingly, a point-like particle possessing mass is also called gravitoelectric monopole. This, therefore, implies that the concept of mass is of no essential significance for gravitomagnetic matter; it is of interest to investigate the motion of gravitomagnetic monopole in curved spacetime. Although Ricci identity is violated due to the non-analytic properties caused by the existence of gravitomagnetic charge, Bianchi identity still holds in the presence of gravitomagnetic charge. It follows that the covariant divergence of $\tilde{G}^{\mu\nu}$ vanishes, namely,

$$\tilde{G}^{\mu\nu}{}_{;\nu} = 0. \tag{3.1}$$

Then in terms of the following field equation

$$\tilde{G}^{\mu\nu} = S^{\mu\nu} \tag{3.2}$$

with the antisymmetric source tensor of gravitomagnetic matter $S^{\mu\nu}$ being $\kappa_1 K^{\mu\nu} + \kappa_2 H^{\mu\nu}$, one can arrive at

$$S^{\mu\nu}{}_{;\nu} = 0 \tag{3.3}$$

which can be regarded as the equation of motion of gravitomagnetic charge in the curved spacetime. It is useful to obtain the low-motion and weak-field-approximation form of Eq. (3.3), which enables us to guarantee that Eq. (3.3) is indeed the equation of motion of gravitomagnetic monopole.

The general Dirac matrices under the weak-field approximation may be obtained via the relations $\gamma_\mu\gamma_v+\gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ and the results are given as follows

$$\gamma^0 = (1+g^0)\beta, \quad \gamma^i = g^i\beta + \gamma^i_{\mathrm{M}}, \tag{3.4}$$

where $i = 1,2,3$; $\beta = \gamma^0_{\mathrm{M}}$. γ^0_{M} and γ^i_{M} are the constant Dirac matrices in the flat Minkowski spacetime. The gravitoelectric potential is defined to be $g^0 = \frac{g^{00}-1}{2}$, and gravitomagnetic vector potentials are $g^i = g^{0i} \quad (i = 1,2,3)$. In the framework of dynamics of point-like particle, the source tensor is therefore rewritten as

$$S^{\mu\nu} = \rho\left[\kappa_1(g^\mu U^\nu - g^\nu U^\mu) + \kappa_2\epsilon^{\mu\nu\alpha\beta}(g_\alpha U_\beta - g_\beta U_\alpha)\right], \tag{3.5}$$

where ρ denotes the density of gravitomagnetic matter. It follows from Eq. (3.3) and Eq. (3.5) that there exists the gravitational Lorentz force density in the expression for the force acting on the gravitomagnetic charge, namely, by ignoring some small term and using the static condition $\frac{\partial}{\partial x^0}g^0 = 0$, one can obtain the following expression (in the unit $c = 1$)

$$\begin{aligned}\kappa_1 g^0\frac{\partial}{\partial x^0}\mathbf{v} = {} & 2\kappa_2\left[\nabla\times\mathbf{g} - \mathbf{v}\times\left(\nabla g^0 - \frac{\partial}{\partial x^0}\mathbf{g}\right)\right] - 2\kappa_2\mathbf{g}\times\left(\frac{\partial}{\partial x^0}\mathbf{v} + \nabla g^0\right) \\ & + \kappa_1 g^0\frac{\partial}{\partial x^0}\mathbf{g} - 2\kappa_2 g^0\left(\nabla g^0\times\mathbf{v}\right)\end{aligned} \tag{3.6}$$

with $\mathbf{v}$ being the velocity of the tested gravitomagnetic monopole. It is apparent that $\nabla\times\mathbf{g} - \mathbf{v}\times(\nabla g^0 - \frac{\partial}{\partial x^0}\mathbf{g})$ is the expression associated with gravitational Lorentz force density. Note that in Eq. (3.6) κ_1, κ_2 are considered coupling constants. However, further analysis shows that at least one of them is not a constant and if the relation

$$\kappa_1 g^0 = 2\kappa_2 \tag{3.7}$$

between them is assumed, then Eq. (3.6) may be rewritten as

$$\frac{\partial}{\partial x^0}\mathbf{v} = \left[\nabla\times\mathbf{g} - \mathbf{v}\times\left(\nabla g^0 - \frac{\partial}{\partial x^0}\mathbf{g}\right)\right], \tag{3.8}$$

where we ignore the small term of second order and the derivative term of coupling coefficients with respect to spacetime coordinate, x^μ. It is well known that the form of Eq. (3.8) is the equation of motion of a particle acted upon by the Lorentz force. Hence, Eq. (3.3) is believed to be the generally relativistic equation of motion of gravitomagnetic monopole in the Riemann spacetime.

Investigation of relativistic dynamics of the topological dual mass is of interest. It should be noted that although gravitomagnetic monopole does not possess mass, it still has energy. Since the dual mass is a kind of topological charge which is very different from Noether charge, no mechanism of interaction may turn it into mass (gravitoelectric charge), and vice versa.

The gravitational analogue of Meissner effect in superconductivity is gravitational Meissner effect. Due to the conservation law of momentum, mass-current density may be conserved in the process of scattering in perfect fluid, which is analogous to the superconductivity of superconducting electrons in superconductors cooled below T_c. Since gravitational field equation under linear approximation is similar to the London′s equations of superconductivity, one can predict that gravitational Meissner effect arises in perfect fluid. The author holds that the investigation of both the effect of gravitomagnetic matter and gravitational Meissner effect may provide us with a valuable insight into the problem of cosmological constant and vacuum gravity [30–33]: the gravitoelectric field (Newtonian field of gravity) produced by the gravitoelectric charge (mass) of the vacuum quantum fluctuations is exactly cancelled by the

gravitoelectric field due to the induced current of the gravitomagnetic charge of the vacuum quantum fluctuations; the gravitomagnetic field produced by the gravitomagnetic charge (dual mass) of the vacuum quantum fluctuations is exactly cancelled by the gravitomagnetic field due to the induced current of the gravitoelectric charge (mass current) of the vacuum quantum fluctuations. Thus, at least in the framework of weak-field approximation, the extreme space-time curvature of vacuum caused by its large energy density does not arise, and the gravitational effects of cosmological constant is eliminated by the contributions of the gravitomagnetic charge (dual mass). If gravitational Meissner effect is of really physical significance, then it is necessary to apply this effect to the early universe. Some related topics such as gravitational Hall effect and gravitational magnetohydrodynamics may be considered for further consideration.

IV. STATIC CYLINDRICALLY SYMMETRIC EXACT SOLUTION OF FIELD EQUATION

In this section the static cylindrically symmetric gravitomagnetic field and the evolution of wavefunction of photon in gravitomagnetic field are considered. Suppose that the form of linear element describing the static cylindrically symmetric gravitomagnetic field is given by

$$\mathrm{d}s^2=\mathrm{d}(x^0)^2-\mathrm{d}x^2-\mathrm{d}y^2-\mathrm{d}z^2+2g_{0x}(y)\mathrm{d}x^0\mathrm{d}x+2g_{0y}(x)\mathrm{d}x^0\mathrm{d}y, \tag{4.1}$$

where we assume that the gravitomagnetic potentials g_{0x} and g_{0y} are functions with respect to y and x, respectively. Thus we obtain all the only nonvanishing values of Christoffel symbols as follows:

$$\Gamma_{0,xy}=\Gamma_{0,yx}=\frac{1}{2}\left(\frac{\partial g_{0x}}{\partial y}+\frac{\partial g_{0y}}{\partial x}\right),\quad \Gamma_{x,0y}=\Gamma_{x,y0}=-\Gamma_{y,0x}=-\Gamma_{y,x0}=\frac{1}{2}\left(\frac{\partial g_{0x}}{\partial y}-\frac{\partial g_{0y}}{\partial x}\right). \tag{4.2}$$

Since the field equation of gravitomagnetic matter is the antisymmetric equation, we might as well take into account a simple case of the following simplified equation

$$\epsilon^{0\alpha\beta\gamma}R^{0}{}_{\alpha\beta\gamma}=\rho_M \tag{4.3}$$

with ρ_M being the parameter associated with the coupling constants and gravitomagnetic charge. It is therefore apparent that Eq. (4.3) agrees with Eq. (3.2). Hence, the solution of the former equation also satisfies the latter. For the reason of the completely antisymmetric property of the Levi-Civita tensor, the contravariant indices α,β,γ should be respectively taken to be x,y,z, namely, we have

$$\epsilon^{0\alpha\beta\gamma}R^{0}{}_{\alpha\beta\gamma}=2\epsilon^{0xyz}(R^{0}{}_{xyz}+R^{0}{}_{zxy}+R^{0}{}_{yzx}). \tag{4.4}$$

There exist the products of two Christoffel symbols, *i.e.*, $g^{\sigma\tau}(\Gamma_{\tau,\alpha\gamma}\Gamma_{\lambda,\sigma\beta}-\Gamma_{\tau,\alpha\beta}\Gamma_{\lambda,\sigma\gamma})$ in the definition of the Riemann curvature, $R_{\lambda\alpha\beta\gamma}$. Apparently, the products of two Christoffel symbols (*i.e.*, the nonlinear terms of field equation) contain the total indices, x,y,z of three-dimensional space coordinate (namely, these indices are taken the permutations of x,y,z) and therefore vanish, in the light of the fact that the Christoffel symbol with index z is vanishing according to Eq. (4.2).

In view of the above discussion, one can conclude that Eq. (4.3) can be exactly reduced to a linear equation. It is easily verified that $R_{\lambda\alpha\beta\gamma}$ ($\lambda=x,y,z$) vanishes with the help of the linear expression for $R_{\lambda\alpha\beta\gamma}$ given by $R_{\lambda\alpha\beta\gamma}=\frac{1}{2}\left(\frac{\partial^2 g_{\lambda\gamma}}{\partial x^\alpha\partial x^\beta}+\frac{\partial^2 g_{\alpha\beta}}{\partial x^\lambda\partial x^\gamma}-\frac{\partial^2 g_{\lambda\beta}}{\partial x^\alpha\partial x^\gamma}-\frac{\partial^2 g_{\alpha\gamma}}{\partial x^\lambda\partial x^\beta}\right)$ and the linear element expressed by Eq. (4.1). We thus obtain that $R^{0}{}_{\alpha\beta\gamma}=g^{00}R_{0\alpha\beta\gamma}$. By the aid of the following expression

$$R_{0\alpha\beta\gamma}=\frac{1}{2}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial g_{0\gamma}}{\partial x^\beta}-\frac{\partial g_{0\beta}}{\partial x^\gamma}\right), \tag{4.5}$$

one can arrive at

$$\epsilon^{0\alpha\beta\gamma}R^{0}{}_{\alpha\beta\gamma}=-\frac{g^{00}}{\sqrt{-g}}\nabla\cdot(\nabla\times\mathbf{g}), \tag{4.6}$$

where the gravitomagnetic vector potentials, $\mathbf{g}$, are defined to be $\mathbf{g}=(-g_{0x},-g_{0y},-g_{0z})$. Substitution of Eq. (4.6) into Eq. (4.3) yields

$$\nabla\cdot(\nabla\times\mathbf{g})=-\frac{\sqrt{-g}}{g^{00}}\rho_M. \tag{4.7}$$

Note that Eq. (4.7) is the exact static gravitational field equation of gravitomagnetic matter derived from Eq. (3.2), where use is made of the expression (4.1) of linear element. For the case of cylindrically symmetric gravitomagnetic field with the nonvanishing ρ_M being present only in the x-y plane with $z=0$, the gravitomagnetic field, $\mathbf{B}_g$, which is defined as $\nabla\times\mathbf{g}$, may be written

$$(\mathbf{B}_g)_z=-\frac{\sigma_M}{2}\left(\frac{\sqrt{-g}}{g^{00}}\right)_{z=0} \tag{4.8}$$

with σ_M being the surface density associated with ρ_M and $\left(\frac{\sqrt{-g}}{g^{00}}\right)_{z=0}$ denoting the value of $\frac{\sqrt{-g}}{g^{00}}$ in the x-y plane where $z=0$. It follows from Eq. (4.8) that the direction of $\mathbf{B}_g$ is parallel to the z-axis. The metric components, g_{0x}, g_{0y}, are therefore readily obtained as follows

$$g_{0x}=\frac{B_g}{2}y, \quad g_{0y}=-\frac{B_g}{2}x, \tag{4.9}$$

with $B_g=-\frac{\sigma_M}{2}\left(\frac{\sqrt{-g}}{g^{00}}\right)_{z=0}$.

In order to obtain the contravariant metric $g^{\mu\nu}$, we calculate the inverse matrix of the metric $(g_{\mu\nu})$ and the result is given as follows

$$(g^{\mu\nu})=\frac{1}{1+g_{0x}^2+g_{0y}^2}\left(\begin{array}{cccc} 1 & g_{0x} & g_{0y} & 0\\ g_{0x} & -(1+g_{0y}^2) & g_{0x}g_{0y} & 0\\ g_{0y} & g_{0x}g_{0y} & -(1+g_{0x}^2) & 0\\ 0 & 0 & 0 & -(1+g_{0x}^2+g_{0y}^2)\end{array}\right). \tag{4.10}$$

It is well known that a photon propagating inside the noncoplanarly curved optical fiber that is wound smoothly on a large enough diameter [34–36] is acted upon by an effective Lorentz force which may be expressed as (in the unit $\hbar=c=1$) [37]

$$\mathbf{f}\equiv\dot{\mathbf{k}}=\mathbf{k}\times\left(\frac{\dot{\mathbf{k}}\times\mathbf{k}}{k^2}\right) \tag{4.11}$$

where the effective magnetic field is $\frac{\dot{\mathbf{k}}\times\mathbf{k}}{k^2}$ with dot denoting the time rate of change of $\mathbf{k}(t)$. Eq. (3.8) has been shown to be the expression for the gravitational force acting on the gravitomagnetic monopole, in the similar manner, one can consider the motion of gravitoelectric charge, for instance, a photon propagating in the static cylindrically symmetric gravitomagnetic field. For the sake of analyzing this problem conveniently, we first take into account the time evolution of wavefunction of a photon in the weak gravitational field. The infinitesimal rotation operator of wavefunction of the photon in gravitomagnetic field is given by $U_R=1-i\Delta\vartheta\cdot\mathbf{J}$ with [26]

$$\Delta\vartheta=\frac{\mathbf{k}(t)\times[\mathbf{k}(t)+\Delta\mathbf{k}]}{k^2}=\frac{\mathbf{k}(t)\times\dot{\mathbf{k}}(t)}{k^2}\Delta t, \tag{4.12}$$

where $\Delta\mathbf{k}$ is defined to be $\dot{\mathbf{k}}\Delta t$. Simple calculation shows that the effective Hamiltonian is of the form $H_{eff}=\frac{\dot{\mathbf{k}}\times\mathbf{k}}{k^2}\cdot\mathbf{J}$. Given that this infinitesimal rotation of wavefunction of the photon is caused by the gravitomagnetic field, it follows from both Eq. (4.11) and (4.12) that the equation of motion of a photon in gravitomagnetic field may be written in the following form

$$\frac{\dot{\mathbf{k}}\times\mathbf{k}}{k^2}=\nabla\times\mathbf{g}. \tag{4.13}$$

This formulation is readily generalized to the case of massive particle moving in gravitomagnetic field and a number of related topics concerning the matter wave in gravitomagnetic field may be further investigated.

V. A POTENTIAL INTERPRETATION OF ANOMALOUS ATTRACTIVE FORCE ACTING ON PIONEER SPACECRAFTS

Taking the effects of gravitomagnetic charge into consideration is believed to be of essential significance in resolving some problems and paradoxes. An illustrative example that has been briefly discussed is its application to the problem of cosmological constant. Additionally, in 1998, Anderson *et al.* reported that, by ruling out a number of nongravitational potential causes such as the solar radiation pressure, precessional attitude-control maneuvers, nonisotropic thermal radiation, radiation of the spacecraft radio beam and so on, radio metric data from the Pioneer 10/11, Galileo and Ulysses spacecrafts indicate an apparent anomalous, constant, acceleration acting on the spacecraft with a magnitude $\sim 8.5\times\ 10^{-8}\mathrm{cm/s}^2$ directed towards the Sun [25]. Is it the effects of dark matter or a modification of gravity? Unfortunately, neither easily works. It is interesting that, by taking the cosmic mass, $M = 10^{53}$ kg, and cosmic length scale, $R = 10^{26}$ m, our calculation shows that this anomalous acceleration is just equal to the value of gravitational field strength on the cosmic boundary due to the total cosmic mass. This fact leads us to consider a theoretical mechanism to interpret this anomalous phenomenon. The author favors that the gravitational Meissner effect may serve as a potential interpretation. Here we give a rough analysis, which contains only the most important features rather than the precise details of this theoretical explanation. Once gravitomagnetic matter exists in the universe, parallel to London's electrodynamics of superconductivity, gravitational field may give rise to an *effective mass* $m_g = \frac{\hbar}{c^2}\sqrt{8\pi G\rho_m}$ due to the self-induced charge current [38], where ρ_m is the mass density of the universe (For the case of arbitrarily strong gravitational fields, see the interesting work of Visser [39], where he defined the mass term of graviton by introducing a non-dynamical background metric). A constant acceleration, a, may result from the Yukawa potential and can be written as

$$\begin{aligned} a &= \frac{GM}{2}\left(\frac{m_g c}{\hbar}\right)^2 = \frac{GM}{c^2}(4\pi\rho_m G) \\ &= \frac{GM}{c^2 R}\frac{G(4\pi R^3\rho_m)}{R^2} \simeq \frac{GM}{R^2}, \end{aligned} \tag{5.1}$$

where for the universe, use is made of $\frac{GM}{c^2 R} \simeq \mathcal{O}(1), \quad 4\pi R^3\rho_m \simeq M$, which holds when approximate estimation is performed. Note, however, that this is an acceleration of repulsive force directed, roughly speaking, from the center of the universe. By analyzing the NASA's Viking ranging data, Anderson, Laing, Lau *et al.* concluded that the anomalous acceleration does not act on the body of large mass such as the Earth and Mars. If gravitational Meissner effect only affected the gravitating body of large mass or large scale rather than spacecraft, then seen from the Sun or Earth, there exists an added attractive force acting on the spacecraft. In the following, $\mathbf{g}_{\mathrm{free}}$ denotes the free-body acceleration caused by the cosmic mass, and apparently the velocity of the Sun and the spacecraft agrees with the following equation (using approximate analysis)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{\mathrm{sun}} = -\mathbf{g}_{\mathrm{free}} + \mathbf{a}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{\mathrm{spacecraft}} = -\mathbf{g}_{\mathrm{free}}, \tag{5.2}$$

then the velocity of the spacecraft relative to the Sun satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}_{\mathrm{spacecraft}} - \mathbf{v}_{\mathrm{sun}}\right) = -\mathbf{a}, \tag{35}$$

where the acceleration of spacecraft due to the Solar gravitation has been ignored. The negative sign on the right handed side of Eq. (35) shows that this added force gives rise to an anomalous, constant, acceleration directed towards the Sun.

Anderson *et al.* reported that without using the apparent acceleration, *i.e.*, using an independent analysis, the compact high accuracy satellite motion program (CHASMP) shows a steady frequency drift of about -6×10^{-9} Hz/s, or 1.5 Hz over 8 year. This equates to a clock acceleration, a_t of -2.8×10^{-18} s^2/s. If there were a systematic drift in the atomic clocks of the DSN (Deep Space Network) or in the time-reference standard signals, this would appear like a nonuniformity of time; *i.e.*, all clocks would be changing with a constant acceleration. Anderson *et al.* believed that people cannot rule out this possibility without adequate analysis. However, we think that the so-called clock acceleration has no fundamental significance, but favors the existence of the anomalous acceleration acting on these spacecraft, seen from the Sun. The relation between coordinate time $\mathrm{d}x^0$ in the solar system and the proper time $\mathrm{d}\tau$ is

$$\frac{\mathrm{d}x^0}{c\mathrm{d}\tau} = 1 - \frac{\phi_{\mathrm{sun}} + ar}{c^2} \tag{5.3}$$

with $\phi_{\rm sun}$ being the gravitational potential caused by the mass of the Sun and r the radial distance from the Sun. By ignoring $\frac{\phi_{\rm sun}}{c^2}$ that is familiar to us, the added clock acceleration may be written as

$$a_t = \frac{{\rm d}^2 x^0}{c{\rm d}\tau^2} = -\frac{a}{c^2}\frac{{\rm d}r}{{\rm d}\tau} = -\frac{a}{c}, \tag{5.4}$$

where $\frac{{\rm d}r}{{\rm d}\tau}$ is taken to be c, the speed of the radio signals from the spacecrafts. It follows from Eq. (5.4) that

$$a = -ca_t. \tag{5.5}$$

This expression can be proved correct by inserting the experimental values, $a_t = -2.8 \times 10^{-18}$ s^2/s and $a = 8.5\times 10^{-10}$m/s^2, into the above equation.

Although it seems natural to interpret this anomalous gravitational phenomenon, there is still something that deserves further research. Why does not the gravitational Meissner effect explicitly affect the body of small mass? Maybe small-mass flow cannot serve as the self-induced charge current that provides the gravitational field with an effective mass in the field equation. The theoretical resolution of this problem is not very definite at present. However, the sole reason that the above resolution of the anomalous acceleration is somewhat satisfactory lies in that no adjustable parameters exist in this theoretical mechanism. It is one of the most important advantages in the above mechanism of gravitational Meissner effect, compared with some possible theories of modification of gravity [40], which are always involving several parameters that cannot be determined by theory itself. These theories of modification of gravity were applied to the problem of the anomalous acceleration but could not calculate the value of the anomalous acceleration consistent with experiments performed by Anderson *et al.* . In spite of some defects, the theoretical mechanism of gravitational Meissner effect suggested above still gives a valuable insight into the problem of anomalous acceleration acting on the Pioneer spacecrafts.

VI. NON-ANALYTIC PROPERTY OF METRIC AND TOPOLOGICAL GRAVITOMAGNETIC CHARGE

Since the gravitomagnetic charge is topological charge of spacetime, it may lead to the non-analytic property of space-time metric, and result in the violation of Ricci identity. This may be illustrated as follows:

Based on the gravitational field equation (2.11) of gravitomagnetic charge, we calculate the following formula

$$\epsilon_{\alpha\beta}{}^{\mu\nu}\tilde{G}_{\mu\nu} = \epsilon_{\alpha\beta}{}^{\mu\nu}(\kappa_1 K_{\mu\nu} + \kappa_2 H_{\mu\nu}), \tag{6.1}$$

and obtain

$$\begin{aligned} \epsilon_{\alpha\beta}{}^{\mu\nu}(\kappa_1 K_{\mu\nu} + \kappa_2 H_{\mu\nu}) &= \kappa_1 H_{\alpha\beta} + 4\kappa_2 K_{\alpha\beta}, \\ \epsilon_{\alpha\beta}{}^{\mu\nu}\tilde{G}_{\mu\nu} &= 2(R_{\alpha\beta} - R_{\beta\alpha}), \end{aligned} \tag{6.2}$$

or

$$R_{\alpha\beta} - R_{\beta\alpha} = \frac{1}{2}\kappa_1 H_{\alpha\beta} + 2\kappa_2 K_{\alpha\beta}, \tag{6.3}$$

where use is made of $\epsilon_{\alpha\beta\nu\mu}\epsilon^{\lambda\sigma\tau\mu} = g^{[\lambda}_{[\alpha}g^{\sigma}_{\beta}g^{\tau]}_{\nu]}$ with λ, σ, τ and α, β, ν being completely antisymmetric, respectively. It follows from Eq. (2.11) and (6.3) that

$$\begin{aligned} 2\kappa_2(R_{\alpha\beta} - R_{\beta\alpha}) - \kappa_1\tilde{G}_{\alpha\beta} &= (4\kappa_2^2 - \kappa_1^2)K_{\alpha\beta}, \\ 2\kappa_1(R_{\alpha\beta} - R_{\beta\alpha}) - 4\kappa_2\tilde{G}_{\alpha\beta} &= (\kappa_1^2 - 4\kappa_2^2)H_{\alpha\beta}. \end{aligned} \tag{6.4}$$

It is known that $R_{\alpha\beta} = R_{\beta\alpha}$ holds in the absence of gravitomagnetic charge in general relativity. However, when there exists gravitomagnetic charge, the symmetric property of Ricci tensor is no longer valid. Further calculation shows that

$$R_{\alpha\beta} - R_{\beta\alpha} = \frac{1}{2}g^{\sigma\tau}\left(\frac{\partial^2 g_{\sigma\tau}}{\partial x^\beta \partial x^\alpha} - \frac{\partial^2 g_{\sigma\tau}}{\partial x^\alpha \partial x^\beta}\right), \tag{6.5}$$

and then it is apparently seen that the non-analytic property of metric leads to the non-integral condition, and that the above case is in exact analogy with that of the Aharonov-Bohm effect in electrodynamics. This, therefore, implies that gravitomagnetic charge possesses the topological or global properties, and that the geometric and non-analytic properties of spacetime deserve detailed study.

VII. CONCLUDING REMARKS

The present paper considers the concept of gravitomagnetic charge and its dynamics in curved spacetime, containing the field equation and the equation of motion and the static exact solution. Differing from the symmetric property (with respect to the space-time coordinate) of gravitational field equation of gravitoelectric matter, the field equation of gravitomagnetic matter possesses the antisymmetric property. This, therefore, implies that the number of the non-analytic metric functions is no more than 6. Although we have no observational evidences for the existence of gravitomagnetic charge, it is still of theoretical significance to investigate the gravity theory of the topological dual mass. It is interesting to speculate on the possibility that the formation mechanism of gravitomagnetic charge may be found in the theory of gravitational interaction, just as the case of the magnetic monopole in various gauge interactions. Once it is present in universe indeed, gravitomagnetic charge leads to physically interesting consequences in astrophysics and cosmology, particularly in the physics of early universe. In the gravitational fields where gravitomagnetic charge exists, the influence of gravitational Meissner effect on the cosmological evolution should not be neglected. In the regime of strong gravity such as the stages of quantum cosmology and inflationary model, gravitational Meissner effect may cause important effects. From the point of view of the framework of classical field theory, matter may be classified into two categories: *gravitomagnetic matter* and *gravitoelectric matter*, according to their different gravitational features[1]. Although it is the classical solution to the field equation, this kind of topological gravitomagnetic monopoles may arise not as fundamental entities in gravity theory.

With foreseeable improvements in detecting and measuring technology, it is possible for us to investigate quantum mechanics in weak-gravitational fields [41,42] associated with gravitomagnetic fields, and the investigation of dual mass is therefore essential in gravity theory. A potential application of gravitational Meissner effect to the problem of the anomalous acceleration is of interest. Although there exists the curiosity as to whether the gravitational Meissner effect is universal for all the gravitating body, regardless of its mass and scale, it gives the theoretical value of anomalous acceleration which is somewhat consistent with the experimental value. This theoretical possibility is of use, given that we have no plausible explanation so far. It is apparently worthwhile to improve this mechanism based on the gravitational Meissner effect.

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[1] Kleinert, H., Gen. Rel. Grav. **32,** 1271 (2000).
[2] Shen, J. Q., Zhu, H. Y. and Li, J., Acta Phys. Sini. **50**,1884 (2001).
[3] Shen, J. Q., Zhu, H. Y., Shi, S. L. and Li, J., Phys. Scr. **65**, 465 (2002).
[4] Aharonov, Y. and Bohm, D., Phys. Rev. **115,** 485 (1959).
[5] Aharonov, Y. and Carmi, G., Found. Phys. **3**, 493 (1959).
[6] Anandan, J., J. Phys. Rev. D **15**, 1448 (1977).
[7] Dresden, M. and Yang, C. N., Phys. Rev. D **20**, 1846 (1979).
[8] Mashhoon, B., Gen. Rel. Grav. **31**, 681 (1999).
[9] Mashhoon, B., Class. Quant. Grav. **17**, 2399 (2000).
[10] Ellis, G. F. R. and Hogan, P. A., Gen. Rel. Grav. **29,** 235 (1997).
[11] Ingraham, R. L., Gen. Rel. Grav. **29**,117 (1997).
[12] Dirac, P. A. M., Proc. Roy. Soc. (London) A **133**, 60 (1931).
[13] Schwinger, J., Phys. Rev. **144**, 1087 (1966).
[14] Yang, C. N., Phys. Rev. D **1**, 2360 (1970).
[15] Yang, C. N., Phys. Rev. Lett. **33**, 445 (1974).
[16] Hooft, G. 't, Nucl. Phys. B **79**, 276 (1974).
[17] Tchrakian, D. H., and Zimmerschied, F., Phys. Rev. D **62**, 045002-1 (2000).
[18] Taub, A. H., Ann. Math. **53**, 472 (1951).
[19] Newman, E. T., Tamburino, L. and Unti, T., J. Math. Phys. **4**, 915 (1963).

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[1] The latter is referred to as the *ordinary matter* and the former can therefore be referred to as the *dual matter*.

[20] Dowker, J. S. and Roche, J. A., Proc. Phys. Soc. London **92**, 1 (1967).
[21] Nouri-Zonoz, M. and Lynden-Bell. D., Class. Quant. Grav. **14**, 3123 (1997).
[22] Dowker, J. S., Gen. Rel. Grav. **5**, 603 (1974).
[23] Zee, A., Phys. Rev. Lett. **55**, 2379 (1985).
[24] Lynden-Bell. D. and Nouri-Zonoz, M., Rev. Mod. Phys. **70**, 427 (1998).
[25] Anderson, J. D., Laing, P. A., Lau, E. L., *et al*, Phys. Rev. Lett. **81**, 2858 (1998).
[26] Li, H. Z., Global Properties of Simple Physical Systems (China: Shanghai Scientific & Technical Publishers) (1998).
[27] Li, H. Z., Guo, S. H., Xian, D. C. and Wu, Y. S., Acta Phys. Sini. **28**, 549 (1979).
[28] Polyakov, A. M., Phys. Lett. B **59**, 82 (1974).
[29] Polyakov, A. M., Nucl. Phys. B **120**, 249 (1974).
[30] Weinberg, S., Rev. Mod. Phys. **61**, 1 (1989).
[31] Datta, D. P., Gen. Rel. Grav. **27**, 341 (1995).
[32] Alvarenga, F. G. and Lemos, N. A., Gen. Rel. Grav. **30**, 681 (1998).
[33] Capozziello, S. and Lambiase, G., 1999 Gen. Rel. Grav. **31**, 1005
[34] Chiao, R. Y. and Wu, Y. S., Phys. Rev. Lett. **57**, 933 (1986).
[35] Tomita, A. and Chiao, R. Y., Phys. Rev. Lett. **57**, 937 (1986).
[36] Kwiat, P. G. and Chiao, R. Y., Phys. Rev. Lett. **66**, 588 (1991).
[37] Shen, J. Q., Zhu, H. Y. and Shi, S. L., Acta Phys. Sini. **51**, 536 (2002).
[38] Hou, B. Y. and Hou, B. Y., Phys. Ener. Fort. Phys. Nucl. **3**, 255 (1979).
[39] Visser, M., Gen. Rel. Grav. **30**, 1717 (1998).
[40] Nieto, M. M. and Goldman, T., Phys. Rep. **205**, 221 (1991).
[41] Lammerzahl, C., Gen. Rel. Grav. **28**, 1043 (1996).
[42] Alvarez, C. and Mann, R., Gen. Rel. Grav. **29**, 245 (1997).